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Sensitive Observables on Infinite-Dimensional Hilbert Spaces

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Abstract

Let Σ be a physical system consisting of two subsystems, S and T: We prove that there are, in the absence of superselection rules, quantum mechanical observables (called "sensitive"), whose expectation value depends on the type of state vector (first type or second type) describing Σ. This result generalizes a previous one obtained under the restriction that the Hilbert spaces of S and T are two dimensional.

1. *Introduction*

Let a quantum mechanical system Σ be given, consisting of two subsystems, S and T. If H_S and H_T are the Hilbert spaces describing, respectively, the system S and the system T, quantum mechanics prescribes that the Hilbert space H_{ST}, associated with the system Σ, is the tensor product of H_S and H_T:

$$H_{ST} = H_S \otimes H_T$$

with the standard scalar product.

It follows directly from definition of tensor product of two vector spaces[2]

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[2] It is useful for the following to recall the definition of tensor product of two vector spaces: Let E, F, G be three vector spaces on the complex field $\mathbb{C}$; we say that G is the tensor product of E and F (denoted by $E \otimes F$) if there exists a bilinear mapping $\Phi : E \times F \to G$ whose image spans G and such that, for all pairs of linearly independent systems of vectors $\{x_\alpha\}$ and $\{y_\beta\}$ in E and F, respectively, the system of vectors $\{\Phi(x_\alpha, y_\beta)\}$ is linearly independent in G. $E \otimes F$ always exists and is unique (up to isomorphisms). If $(x, y) \in E \times F$ we put $\Phi(x, y) = x \otimes y$; it is trivial that $\Phi(E \times F)$ is not in general, a vector space and therefore $\Phi(E \times F) \neq E \otimes F$. If E and F are Hilbert spaces the standard scalar product on $E \otimes F$ makes continuous the bilinear mapping Φ.

that an element u of H_{ST} cannot be put, in general, in the form $v \otimes w$, where v and w are elements of H_S and H_T, respectively.

Therefore let us introduce the following definition (Capasso *et al.*, 1973):

Definition 1.1. A vector $u \in H_{ST}$ is called vector of the first type if there exist $v \in H_S$, $w \in H_T$ such that $u = v \otimes w$; u is called vector of the second type if it is not a vector of the first type.

Let us put[3]

$$Q = \{x \otimes y \in H_{ST} : x \in H_S, y \in H_T, \|x\| = \|y\| = 1\}$$

Let us denote by $\mathfrak{S}$ a statistical ensemble of identical systems Σ and, introduce the following definitions:

Definition 1.2. $\mathfrak{S}$ is called mixture of the first type if there exists a a family $\{\mathfrak{S}_k\}_{k \in \{1,\ldots,m\}}$, $\mathfrak{S}_k \subset \mathfrak{S}$, $\mathfrak{S}_1 \cup \cdots \cup \mathfrak{S}_m = \mathfrak{S}$, such that for every $k \in \{1, \ldots, m\}$ there exists a vector $u_k \in Q$ describing all systems $\Sigma \in \mathfrak{S}_k$.

Definition 1.3. $\mathfrak{S}$ is called a mixture of the second type if there exists a vector of the second type $u \in H_{ST}$, $\|u\| = 1$, describing all systems $\Sigma \in \mathfrak{S}$.

2. *Sensitive Observables*

We shall prove that a mixture of the second type is observably different from all mixtures of the first type. This statement can be put in a precise form once the following definition is introduced (Capasso *et al.*, 1973).

Definition 2.1. If $\mathfrak{S}$ is a mixture of the second type of systems Σ, an observable Γ of Σ is called sensitive for $\mathfrak{S}$ if $\langle\Gamma\rangle_{\mathfrak{S}} \neq \langle\Gamma\rangle_{\mathfrak{S}'}$ for all mixtures of the first type $\mathfrak{S}'$ of systems Σ.[4]

This definition is meaningful because it can be shown that (Capasso *et al.*, 1973), if $\mathfrak{S}$ is a mixture of the second type of systems Σ, there are observables (of Σ) that are not sensitive.

We shall prove the following theorem, which clarifies the statement we made at the beginning of this section:

[3] If E is an Hilbert space we denote by $(\,|\,)$ its scalar product and by $\|\;\|$ the norm derived from it.

[4] We denote by $\langle\Gamma\rangle_{\mathfrak{S}}$ the expectation value of the observable Γ on the statistical ensemble $\mathfrak{S}$.

Theorem 2.1. Let us suppose that to every Hermitian operator (on H_S, H_T, H_{ST}) is associated one and only one observable[5] (of S, T and Σ, respectively). On this hypothesis we shall prove that, if $\mathfrak{S}$ is a mixture of the second type of systems Σ, there exists an observable of Σ sensitive for $\mathfrak{S}$.

Let us observe that the result contained in this theorem has been obtained by Capasso *et al.* in the paper cited under the hypothesis that H_S and H_T are two dimensional and by the use of Bell's inequality (Bell, 1965).

Let us prove Theorem 2.1.[6]

Let $\mathfrak{S}$ be a mixture of the second type described by the vector of the second type x_0, $\|x_0\| = 1$.

Let us initially prove that

$$(y|P_{x_0}(y)) < 1 \quad \text{for all} \quad y \in Q \tag{2.1}$$

where P_{x_0} is the projector on the subspace $\mathbb{C}x_0$; that is,

$$P_{x_0} : x \in H_{ST} \to P_{x_0}(x) = (x_0|x)x_0$$

If $y \in Q$, by Schwarz inequality:

$$(y|P_{x_0}(y)) = |(y|x_0)|^2 \leqslant \|y\|^2 \cdot \|x_0\|^2 = 1 \tag{2.2}$$

Let us now observe that

$$(y|P_{x_0}(y)) \neq 1 \tag{2.3}$$

in fact, by virtue of (2.2),[7]

$$\begin{aligned}((y|P_{x_0}(y)) = 1) &\Rightarrow (|(y|x_0)|^2 = \|y\|^2 \cdot \|x_0\|^2)\\ &\Rightarrow (\text{there exists } \alpha \in \mathbb{C},\ |\alpha|\\ &= 1 \text{ such that } x_0 = \alpha y) \Rightarrow (x_0 \in Q)\end{aligned}$$

and this is in contrast with the assumption that x_0 is a vector of the second type.

Therefore from (2.2) and (2.3) we deduce that (2.1) holds.

Now let $\mathfrak{S}'$ be a mixture of the first type of N identical systems Σ; therefore, by definition, there exists a covering $\{\mathfrak{S}'_k\}_{k \in \{1,\ldots,m\}}$ of $\mathfrak{S}'$ such that for every $k \in \{1, \ldots, m\}$ there exists $y_k \in Q$ describing the n_k systems $\Sigma \in \mathfrak{S}'_k (\Sigma_k n_k = N)$.

[5] For this reason we shall identify every Hermitian operator with the observable it represents.

[6] This proof still holds if in Definition 1.2 we assume $Q = \{v \in H_{ST} | v \text{ vector of the first type}, \|v\| = 1\}$.

[7] A straight calculation shows that

$$(|(y|x_0)|^2 = \|x_0\|^2 \cdot \|y\|^2) \Rightarrow \left(\left\|x_0 - \frac{(y|x_0)}{\|y\|^2}\, y\right\|^2 = 0\right)$$

By virtue of (2.1), we have

$$\langle P_{x_0}\rangle_{\mathfrak{S}'} = \sum_{k=1}^{m} \frac{n_k}{N} \langle P_{x_0}\rangle_{\mathfrak{S}'_k} = \sum_{k=1}^{m} \frac{n_k}{N} (y_k | P_{x_0}(y_k)) < \sum_{k=1}^{m} \frac{n_k}{N} = 1$$

On the other side we have

$$\langle P_{x_0}\rangle_{\mathfrak{S}} = (x_0 | P_{x_0}(x_0)) = \|x_0\|^2 = 1$$

Therefore, by definition, P_{x_0} is a sensitive observable for $\mathfrak{S}$.

Preserving the notation just introduced, let us observe that, in general, we cannot say that

$$\sigma = \sup_{y \in Q} (y | P_{x_0}(y)) = \sup_{y \in Q} |(y | x_0)|^2 < 1 \tag{2.4}$$

If (2.4) holds, we have

$$\Delta(\mathfrak{S}, \mathfrak{S}') = \langle P_{x_0}\rangle_{\mathfrak{S}} - \langle P_{x_0}\rangle_{\mathfrak{S}'} \geqslant 1 - \sigma > 0 \tag{2.5}$$

for all mixtures of the first type $\mathfrak{S}'$; that is, $\Delta(\mathfrak{S}, \mathfrak{S}')$ cannot be made arbitrary small for suitable choice of $\mathfrak{S}'$.

Therefore it is interesting to search vector states of the second type x_0 such that (2.4) holds.

To this end let us distinguish two cases:

(i) Let H_S and H_T be finite dimensional, then the following theorem holds:

Theorem 2.2. If $x_0 \in H_{ST} - Q$, $\|x_0\| = 1$, there exists $y_0 \in Q$ such that

$$\sup_{y \in Q} |(y | x_0)|^2 = |(y_0 | x_0)|^2$$

Proof. Let us initially prove that Q is *compact*: Let $(x_n \otimes y_n)_{n \in \mathbb{N}}$ be a sequence of elements of the set Q; therefore

$$\forall n \in \mathbb{N} \qquad \|x_n\| = \|y_n\| = 1$$

Then, by Bolzano's theorem, we can select from $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ two subsequences $(x_{nk})_{k \in \mathbb{N}}$, $(y_{nk})_{k \in \mathbb{N}}$ convergent to $x \in H_S$ and $y \in H_T$, respectively. On the other hand, by continuity of the bilinear mapping Φ we have

$$\lim_{k \to \infty} x_{nk} \otimes y_{n_k} = \lim_{k \to \infty} \Phi(x_{nk}, y_{nk}) = \Phi(x, y) = x \otimes y$$

Let us now observe that $\|x\| = \|y\| = 1$, therefore $x \otimes y \in Q$; hence Q is a *compact* set. Thus, by continuity of the functional $f : x \in H_{ST} \to f(x) = |(x|x_0)|^2$, we deduce that there exists $y_0 \in Q$ such that $\sup_{y \in Q} |(y|x_0)|^2 = |(y_0|x_0)|^2$

(ii) Let us suppose that H_S and H_T are infinite dimensional and introduce the following notations: let us denote by E the *convex envelope* of Q, that is,

$$E = \left\{ \sum_i \lambda_i x_i : \lambda_i \geqslant 0, \quad \sum_i \lambda_i = 1 \quad \text{and} \quad x_i \in Q \right\}$$

and by $\bar{E}$ the *closure* of E, that is

$$\bar{E} = \left\{ x \in H_{ST} \mid \exists (x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}} \quad \text{such that} \quad \lim_{n \to \infty} x_n = x \right\}$$

Now we shall prove the following theorem:

Theorem 2.3. If $x_0 \in H_{ST} - \bar{E}$, $\|x_0\| = 1$ we have

$$\sup_{y \in \bar{E}} |(y|x_0)|^2 < 1 \tag{2.6}$$

Proof. The following implications are a straight consequence of the definitions we have given:

$$x \in \bar{E} \Rightarrow \|x\| \leqslant 1 \tag{2.7}$$

$$\left.\begin{matrix} x \in \bar{E} \\ \alpha \in \mathbb{C}, \ |\alpha| = 1 \end{matrix}\right\} \Rightarrow \alpha x \in \bar{E} \tag{2.8}$$

Let us now consider the functional

$$f : y \in \bar{E} \to f(y) = |(y|x_0)|^2$$

where $x_0 \in H_{ST} - \bar{E}$, $\|x_0\| = 1$, and let us prove that

$$\forall y \in \bar{E} : f(y) < 1 \tag{2.9}$$

If $y \in \bar{E}$, $y \neq 0$ [if $y = 0$ obviously $f(y) < 1$], by virtue of (2.7), we have

$$|(y|x_0)|^2 \leqslant \|y\|^2 \ \|x_0\|^2 \leqslant 1 \tag{2.10}$$

Therefore, if $f(y) = |(y|x_0)|^2 = 1$, we obtain

$$|(y|x_0)|^2 = \|y\|^2 \cdot \|x_0\|^2 \quad \text{and} \quad \|y\| = 1$$

from which

$$x_0 = \alpha y \qquad \alpha \in \mathbb{C}, \qquad |\alpha| = 1$$

then, by virtue of (2.8), we obtain

$$x_0 \in \bar{E}$$

in contrast with the hypothesis $x_0 \in H_{ST} - \bar{E}$. Therefore (2.9) holds.

Let us observe that $\bar{E}$ is a bounded set of the Hilbert space H_{ST} and therefore it is a *relatively compact* set[8] for the *weak topology*; moreover $\bar{E}$ is *weakly closed*.[9]

[8] Every bounded set in a Hilbert space is *weakly relatively compact*: that is, its *weak closure* is a *compact* set for the *weak topology*.

[9] Every *strongly closed convex* set in a Hilbert space is *weakly closed*.

Therefore $\bar{E}$ is a *weakly compact* set. On the other hand the functional f is *weakly continuous* and therefore, by virtue of the Weierstrass theorem, we conclude that there exists $y_0 \in \bar{E}$ such that

$$\sup_{y \in \bar{E}} f(y) = f(y_0) \tag{2.11}$$

Hence, by (2.9) and (2.11), the inequality (2.6) holds.

References

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